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On the remarkable spectrum of a non-Hermitian random matrix model

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Abstract
A non-Hermitian random matrix model proposed a few years ago has a remarkably intricate spectrum. Various attempts have been made to understand the spectrum, but even its dimension is not known. Using the Dyson–Schmidt equation, we show that the spectrum consists of a non-denumerable set of lines in the complex plane. Each line is the support of the spectrum of a periodic Hamiltonian, obtained by the infinite repetition of any finite sequence of the disorder variables. Our approach is based on the 'theory of words'. We make a complete study of all four-letter words. The spectrum is complicated because our matrix contains everything that will ever be written in the history of the universe, including this particular paper.

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1. A class of non-Hermitian random matrix models

Some time ago, Feinberg and one of us (in a paper to be referred to as FZ [1]) proposed the study of the equation

$$\psi_{k+1} + r_{k-1}\psi_{k-1} = E\psi_k \tag{1}$$

where the real numbers r_k are generated from some random distribution. Two particularly simple models were studied: (A) the r_k are equal to ± 1 with equal probability, and (B) $r_k = \mathrm{e}^{\mathrm{i}\theta_k}$ with the angle θ_k uniformly distributed between 0 and 2π.

Imposing the boundary condition $\psi_0 = \psi_{N+2} = 0$ on (1) we can write the set of equations as the eigenvalue equation

$$H_N\psi = E\psi \tag{2}$$

with ψ the column eigenvector with components $\{\psi_1, \psi_2, \ldots, \psi_N, \psi_{N+1}\}$ and H_N the $(N+1) \times (N+1)$ non-Hermitian random matrix

$$H_N = \begin{pmatrix} 0 & 1 & 0 & & & \\ r_1 & 0 & 1 & 0 & & \\ 0 & r_2 & 0 & 1 & 0 & \\ & 0 & \ddots & \ddots & \ddots & 0 \\ & & 0 & r_{N-1} & 0 & 1 \\ & & & 0 & r_N & 0 \end{pmatrix}.$$

While quantum mechanics is of course Hermitian it is convenient to think of H as a Hamiltonian and (1) as the non-Hermitian Schrödinger equation describing the propagation of a particle hopping on a one-dimensional lattice.

Some applications of non-Hermitian random Hamiltonians include vortex line pinning in superconductors [2–4] and growth models in population biology [5]. A genuine localization transition can occur for a random non-Hermitian Schrödinger Hamiltonian [6–13] in one dimension.

As mentioned in FZ, with the open chain boundary condition $\psi_0 = \psi_{N+2} = 0$ the more general equation

$$s_{k+1}\psi_{k+1} + r_{k-1}\psi_{k-1} = E\psi_k \tag{3}$$

can always be reduced to (1) by an appropriate 'gauge' transformation $\psi_k \to \lambda_k\psi_k$. Furthermore, applying the transformation $\psi_k \to u^{-k}\psi_k$ to (1) we see that if we change $r_k \to u^2 r_k$ then the spectrum changes by $E \to uE$. Thus, scaling the magnitude of the r_k merely stretches the spectrum, and flipping the sign of all the r_k corresponds to rotating the spectrum by 90°.

It is also useful to formulate the problem in the transfer matrix formalism. Write (1) as

$$\begin{pmatrix} \psi_{k+1} \\ \psi_k \end{pmatrix} = T_{k-1} \begin{pmatrix} \psi_k \\ \psi_{k-1} \end{pmatrix} \tag{4}$$

where the transfer matrix T_k is defined as the 2×2 matrix

$$T_k = \begin{pmatrix} E & -r_k \\ 1 & 0 \end{pmatrix}. \tag{5}$$

Define $P \equiv T_N T_{N-1} \cdots T_2 T_1$. Then the boundary condition implies

$$EP_{11} + P_{12} = 0. \tag{6}$$

The solution of this polynomial equation in E determines the spectrum.

Since H is non-Hermitian the eigenvalues invade the complex plane. For model B, the spectrum has an obvious rotational symmetry and forms a disc (see figure 1, which displays the support of the density of states). An expansion of the density of eigenvalues around $E = 0$ to very high orders in E has been given by Derrida *et al* [14]. This analytic expansion, however, cannot predict singularities in the density of states.

In contrast, for model A the spectrum enjoys only a rectangular $Z_2 \otimes Z_2$ symmetry. The first Z_2 corresponds to $E \to E^*$ obtained by complex conjugating the eigenvalue equation $H\psi = E\psi$. The second Z_2 corresponds to $E \to -E$ obtained by the bipartite transformation $\psi_k \to (-)^k\psi_k$. Remarkably, FZ found that the spectrum has an enormously complicated fractal-like form. In figure 2 we plot the support of the density of eigenvalues in the complex plane for a 4000×4000 matrix, for a specific realization of the disorder. In figure 3 we plot the support of the density of states in the complex plane for a 1000×1000 matrix, averaged over 100 realizations of the disorder.

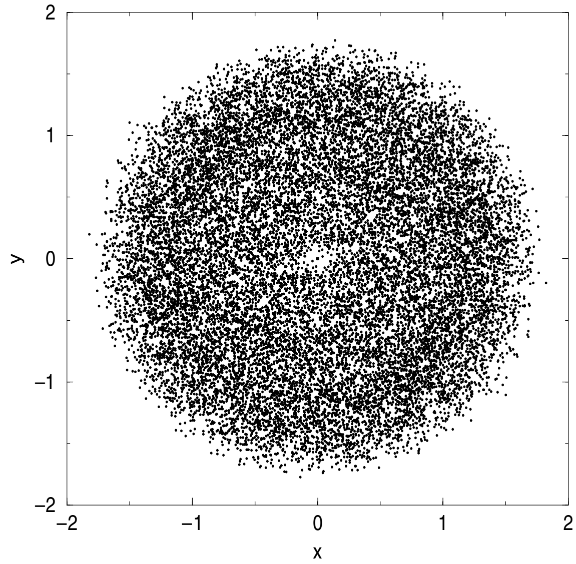

Figure 1. Support of the density of states for model B.

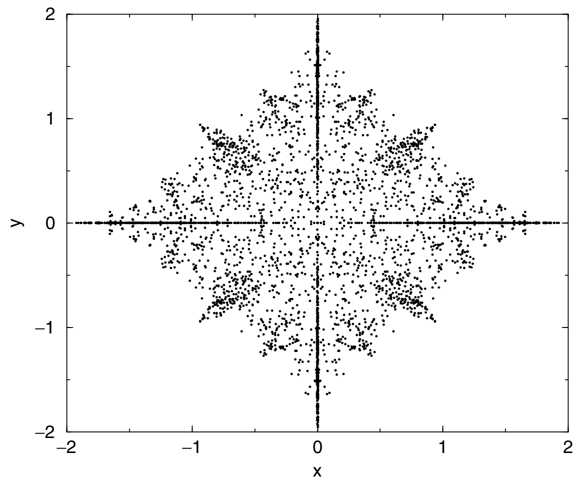

Figure 2. Support of the density of states for model A for a 4000 × 4000 matrix and a single realization of disorder.

Contrasting figures 2 and 3 with figure 1, it can be seen why it has been a challenge for mathematical physicists to understand the nature of the spectrum.

2. Basic formalism

In general, for random non-Hermitian matrices H, the density of eigenvalues can be obtained by

$$\rho(x, y) = \frac{1}{\pi}\frac{\partial}{\partial z^*}G(z, z^*) \tag{7}$$

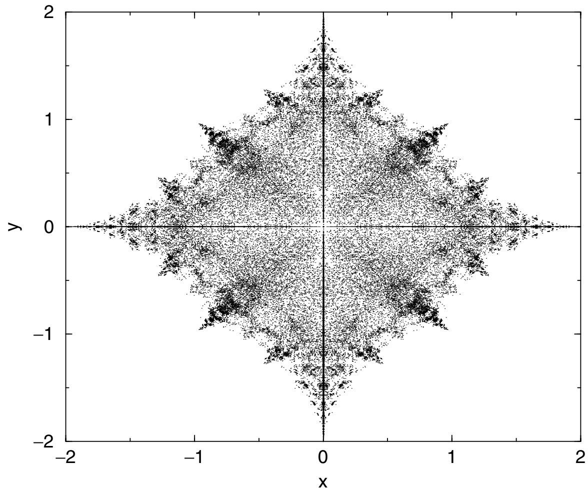

Figure 3. Support of the density of states for model A for a 1000×1000 matrix averaged over 100 realizations of disorder.

where the Green function is defined by

$$G(z, z^*) = \left\langle \frac{1}{N} \mathrm{tr} \, \frac{1}{z - H} \right\rangle$$

with the bracket denoting averaging and $z = x + \mathrm{i}y$ (see, for example, [9] for a proof of these relations). Equation (7) follows from the identity

$$\frac{\partial}{\partial z^*} \frac{\partial}{\partial z} \log z = \frac{\partial}{\partial z^*} \frac{1}{z} = \pi \delta(x) \delta(y) \tag{8}$$

where $z = x + \mathrm{i}y$. Expanding $G(z, z^*) = \frac{1}{z} \sum_{k=0}^{\infty} z^{-k} \left\langle \frac{1}{N} \mathrm{tr} \, H^k \right\rangle$ we see that $\left\langle \frac{1}{N} \mathrm{tr} H^k \right\rangle$ counts the number of paths of a particle returning to the origin in k steps.

Evidently, for model A each link has to be traversed four times. The spectrum of model A was studied by Cicuta *et al* [15] and by Gross and one of us [16] by counting the paths. In particular, Cicuta *et al* gave an explicit expression for the number of paths. Recently, the more general problem of even-visiting random walks has been studied extensively (see [17]). In addition, exact analytic results for the Lyapunov exponent have been found by Sire and Krapivsky [18].

In this paper we propose a different approach based on the theory of words. We will focus on model A although some of our results apply to the general class of models described by (3).

One important issue is the dimensionality of the spectrum of model A. In general, the spectrum of non-Hermitian random matrices when averaged over the randomness is two-dimensional (for example, the spectrum of model B). Many of the authors who have looked at model A believe that its spectrum, as shown in figures 2 and 3, is quasi-zero-dimensional: the spectrum seems to consist of many accumulation points. Here we claim that the dimension of the spectrum actually lies between 1 and 2, in the sense described below.

3. Distribution of the characteristic ratio

Consider the degree $(N+1)$ characteristic polynomial $\Delta_{N+1}(z) \equiv \det(H_N + zI_{N+1})$, where I_{N+1} denotes the $(N+1) \times (N+1)$ identity matrix. We easily obtain the recursion relation

$$\Delta_{N+1}(z) = z\Delta_N(z) - r_N\Delta_{N-1}(z)$$

with $\Delta_0(z) = 1$ and $\Delta_1(z) = z$. Note that this second-order recursion relation can be expressed in terms of transfer matrices as

$$\begin{pmatrix} \Delta_{N+1}(z) \\ \Delta_N(z) \end{pmatrix} = \begin{pmatrix} z & -r_N \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta_N(z) \\ \Delta_{N-1}(z) \end{pmatrix} \tag{9}$$

very similar to those defined for the wavefunction ψ, with the transfer matrix given by

$$U_N = \begin{pmatrix} z & -r_N \\ 1 & 0 \end{pmatrix}. \tag{10}$$

Following Dyson and Schmidt [19, 20] we consider the characteristic ratio

$$y_k(z) \equiv \frac{\Delta_k(z)}{\Delta_{k-1}(z)} \tag{11}$$

which satisfies the recursion relation

$$y_{k+1} = z - \frac{r_k}{y_k} \tag{12}$$

with initial condition $y_1 = z$.

The hope is that, while the characteristic polynomials $\Delta_k(z)$ obviously change dramatically as k varies, the characteristic ratio $y_k(z)$ might converge asymptotically.

From the definition in equation (11), it is clear that a point z is in the spectrum of the $N \times N$ matrix iff $y_N(z) = 0$ and $y_{N+1}(z) = \infty$. Thus a point z belongs to the spectrum if the corresponding set of variables $\{y_N\}$ is unbounded, that is, if the probability of escape of the variable y_n to ∞ is finite. As shown in [17], this condition is sufficient to determine the spectrum along the real axis ($z \in R$), but is insufficient in the complex case.

Let $P_k(y_k)$ be the probability distribution of y_k (note that y_k is complex). Then

$$P_{k+1}(y_{k+1}) = \left\langle \int \mathrm{d}^2 y_k\, P_k(y_k)\delta^{(2)}\left(y_{k+1} - z + \frac{r_k}{y_k}\right)\right\rangle$$

where the brackets denote the average over the disorder variables $\{r_k\}$. In the thermodynamic limit $k \to \infty$, under fairly general conditions [21] it can be shown that the probability distribution has a limit $P(y)$, called the invariant distribution, which is determined by the self-consistent equation

$$\begin{aligned} P(y) &= \left\langle \int \mathrm{d}^2 t\, P(t)\delta^{(2)}\left(y - z + \frac{r}{t}\right)\right\rangle \\ &= \left\langle \frac{|r|^2}{|z-y|^4} P\left(\frac{r}{z-y}\right)\right\rangle \end{aligned}$$

where we have used the fact that $\delta^{(2)}(f(z))$ near a zero, z_0, of $f(z)$ is given by $\delta^{(2)}(z - z_0)\left|\frac{\mathrm{d}f}{\mathrm{d}z}\right|^{-2}$.

For model A, we obtain the amusing equation

$$P(y) = \frac{1}{2|z-y|^4}\left(P\left(\frac{1}{z-y}\right) + P\left(\frac{-1}{z-y}\right)\right). \tag{13}$$

This type of equation has been studied extensively for the real case in [22–24]. It can be shown that it can be solved by the ansatz

$$P(y) = \sum_j a_j \delta^{(2)}(y - b_j(z)) \tag{14}$$

where the b_j depend on z whereas the a_j do not. Note that the index j is not necessarily an integer and can refer to a continuous set. In addition, it can be shown that the $b_j(z)$ are the stable fixed points of the product of any sequence of transfer matrices U (see the next section).

Plugging (14) into (13) we have

$$\sum_j a_j \delta^{(2)}(y - b_j) = \frac{1}{2}\sum_j a_j \left\{\delta^{(2)}\left(y - z + \frac{1}{b_j}\right) + \delta^{(2)}\left(y - z - \frac{1}{b_j}\right)\right\}.$$

Since the right-hand side has twice as many delta function spikes as the left-hand side, for the two sides to match we expect that, in general, the index j would have to run over an infinite set.

For a given complex number z, we demand that the two sets of complex numbers $\{b_j\}$ and $\{z - 1/b_j, z + 1/b_j\}$ be the same. This very stringent condition should then determine the $\{b_j(z)\}$.

To see how this works, focus on a specific b_1 (since the label j has not been specified this can represent any b_j). It is equal to either $z - 1/b_2$ or $z + 1/b_2$ for some b_2. But b_2 must in its turn be equal to either $z - 1/b_3$ or $z + 1/b_3$ for some b_3. This process of identification must continue until we return to b_1. Indeed, if the process of returning to b_1 occurs in a finite number of steps L, then it will repeat indefinitely (since the system is back at its starting point b_1). It is this infinite repetition which gives a finite weight to the δ-function at b_1. By contrast, if the number of steps needed to return to the initial point b_1 is infinite, then the weight associated with this point vanishes and it will not be present in the spectrum.

We thus conclude that the support of the distribution of $P(y)$ is the closure of the set of all the stable fixed points of the product of any sequence of transfer matrices U. We also conclude that the support of the density of states of the non-Hermitian matrix, in the thermodynamic limit, is given by the zeros of any stable fixed point: $b_j(z) = 0$. What is important to note is that the a_j are independent of z and depend only on the length of the word. Thus, we conclude that the set of complex numbers $\{b_j(z)\}$ is determined by the solution of an infinite number of fixed point equations.

4. The theory of words

It is useful here to introduce the theory of words. A word w of length L is defined as the sequence $\{w_1, w_2, \ldots, w_L\}$ where the letters $w_j = \pm 1$. In other words, we have a binary alphabet. Let us also define the repetition of a given word a specific number of times as a simple sentence. We can then string together simple sentences to form paragraphs.

For a given word w of length L, consider a function $f_L(b; z, w)$ to be constructed iteratively. For notational simplicity we will suppress the dependence of f_L on b, z and w, indicating only its dependence on the length L of the word w. The iteration begins with

$$f_1 = z - \frac{w_1}{b}$$

and continues with

$$f_{j+1} = z - \frac{w_{j+1}}{f_j}. \tag{15}$$

We define $f(b; z, w) \equiv f_L$.

The set of complex numbers $\{b_j(z)\}$ is then determined as follows. Consider the set of all possible words. For each word w, determine the solution of the fixed point equation

$$b = f(b; z, w).$$

By considering small deviations from the solution, we see that the solution is a stable fixed point only if

$$\left|\frac{\partial f(b; z, w)}{\partial b}\right| < 1. \tag{16}$$

The set of all possible words w generates the set of complex numbers $\{b_j(z)\}$. In other words, b is determined by a continued fraction equation, since

$$f(b; z, w) = z - \cfrac{w_L}{z - \cfrac{w_{L-1}}{z - \cfrac{w_{L-2}}{\ddots\, \cfrac{}{z - \cfrac{w_1}{b}}}}}.$$

We see that f_j has the form

$$f_j = \frac{\alpha_j b + \beta_j}{\alpha_{j-1} b + \beta_{j-1}} \tag{17}$$

with the polynomials α_j and β_j determined by the recursion relations

$$\alpha_{j+1} = z\alpha_j - w_{j+1}\alpha_{j-1} \tag{18}$$

and

$$\beta_{j+1} = z\beta_j - w_{j+1}\beta_{j-1} \tag{19}$$

with the initial condition $\alpha_0 = 1, \alpha_1 = z, \beta_0 = 0$ and $\beta_1 = -w_1$. We note that α_j and β_j satisfy the same recursion relation as that satisfied by Δ_j with the correspondence $w_{j+1} \leftrightarrow r_j$. Note also that (18) and (19) can be packaged as the matrix equation

$$\begin{pmatrix} \alpha_{j+1} & \beta_{j+1} \\ \alpha_j & \beta_j \end{pmatrix} = \begin{pmatrix} z & -w_{j+1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_j & \beta_j \\ \alpha_{j-1} & \beta_{j-1} \end{pmatrix} \tag{20}$$

where the transfer matrix U_j defined in the previous section appears. This is closely related to the transfer matrix formalism discussed earlier. Indeed, defining $W_j \equiv \left(\begin{smallmatrix} \alpha_j & \beta_j \\ \alpha_{j-1} & \beta_{j-1} \end{smallmatrix}\right)$ we have the initial condition $W_1 = \left(\begin{smallmatrix} z & -w_1 \\ 1 & 0 \end{smallmatrix}\right)$. Hence a given word w of length L can also be characterized by a matrix

$$W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{21}$$

where for convenience we have written $\alpha = \alpha_L, \beta = \beta_L, \gamma = \alpha_{L-1}$ and $\delta = \beta_{L-1}$.

For a given word w, the fixed point value b is determined by the quadratic equation

$$b = \frac{\alpha b + \beta}{\gamma b + \delta} \tag{22}$$

which is the fixed point equation of the homographic mapping associated with the matrix M. The geometric interpretation is clear: the matrix W acts on two-component vectors v, and we ask for the set of v such that the ratio b of the first component to the second component is left invariant by the transformation. In other words, we look for the projective space left invariant by the transformation W: the fixed point value b defines the direction of the invariant ray.

Hence b is given by

$$b = \frac{P(z) \pm \sqrt{Q(z)}}{2R(z)} \tag{23}$$

with P, Q, R polynomials of degree $2L$ in z, where L denotes the length of the word w. Explicitly,

$$P_L(z) = \alpha_L - \beta_{L-1} \tag{24}$$

$$Q_L(z) = (\alpha_L - \beta_{L-1})^2 + 4\alpha_{L-1}\beta_L \tag{25}$$

and

$$R_L(z) = \alpha_{L-1}. \tag{26}$$

The stability condition (16) determines which root of (23) is to be chosen.

We will see shortly that $b(z)$ determines the spectrum. Anticipating this, we see that if we form a compound word by stringing the word w together twice (for example, the Japanese word 'nurunuru') then we expect the contribution to the spectrum to be the same. But given the preceding discussion, this is obvious, since if a ray is left invariant by W, it is manifestly left invariant by W^2.

5. Density of eigenvalues

Once we have determined $b_j(z)$, how do we extract the density of eigenvalues? The eigenvalues $\{\lambda_i\}$ of the matrix H are given by $\Delta_N(z) = \Pi_{i=1}^N (z - \lambda_i)$. From (11) we have $\sum_{k=1}^N \log y_k \simeq \log \Delta_N(z) = \sum_{i=1}^N \log(z - \lambda_i)$, and thus

$$\int \mathrm{d}^2 y\, P(y) \log y = \left\langle \frac{1}{N} \sum_{i=1}^{N} \log(z - \lambda_i) \right\rangle. \tag{27}$$

Using the identity (8) we can differentiate the right-hand side of (27) to obtain the density of eigenvalues in the complex plane

$$\rho = \frac{1}{\pi} \frac{\partial}{\partial z^*} \frac{\partial}{\partial z} \int \mathrm{d}^2 y\, P(y) \log y.$$

Plugging in our solution

$$P(y) = \sum_j a_j \delta^{(2)}(y - b_j)$$

we finally deduce that $\rho = \frac{1}{\pi} \frac{\partial}{\partial z^*} \frac{\partial}{\partial z} \sum_j a_j \log b_j = \frac{1}{\pi} \frac{\partial}{\partial z^*} \sum_j a_j \frac{1}{b_j} \frac{\partial b_j}{\partial z}$.

Since the a_j do not depend on z, the spectrum is determined by the zeros of the fixed point solutions $b_j(z)$.

We see from (23) that the density of eigenvalues is given as a sum over j of terms such as

$$\frac{\partial}{\partial z^*} \left\{ \frac{1}{P(z) \pm \sqrt{Q(z)}} \left[P'(z) \pm \frac{Q'(z)}{2\sqrt{Q(z)}} \right] - \frac{R'(z)}{R(z)} \right\}.$$

Thus the spectrum consists of isolated poles given by the zeros of R and $P(z) \pm \sqrt{Q(z)}$, and of the cuts of $\sqrt{Q(z)}$, and is made of isolated points plus curved line segments connecting the zeros of $Q(z)$.

Contrary to what some authors have believed, the spectrum is not zero-dimensional, but $(0 + 1 + \delta)$-dimensional, with $\delta \leqslant 1$. Each word gives rise to a line segment, and words which

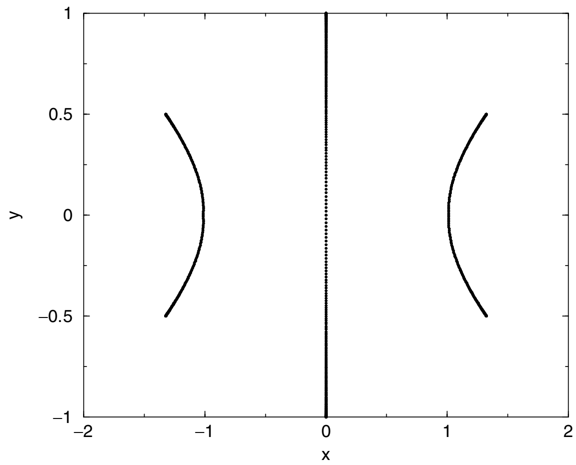

Figure 4. Support of the density of states of the periodic word {+ + −}.

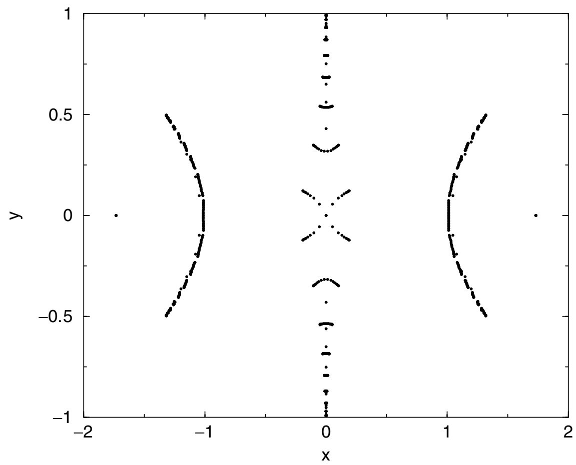

Figure 5. Support of the density of states of the periodic word {+ + −} *corrupted*.

differ slightly from each other give rise to line segments near each other. Indeed, given a word w, it is possible to construct a word w' with a spectrum as close to that of w as desired. For that purpose, we may construct w' as $w' = w^{l_1} v w^{l_2}$ where v is any 'corrupting' word, and the two lengths l_1 and l_2 are sufficiently long. Indeed, in terms of transfer matrices and invariant rays, we see that w^{l_2} acting on any initial ray brings it close to the stable invariant ray of w. Then the direction of this ray is corrupted by v, but it is brought back arbitrarily close to the invariant ray of w by applying the transfer matrix w^{l_1}, provided that l_1 is large enough. Presumably (although this remains to be proved rigorously), the spectrum associated with the corrupted word w' can be made as close as we want to that of w. We thus have this property that for any word w, there is a word w' generating a spectrum as close as we want to that of w. In figures 4 and 5 we plot the eigenstates of a word $w = \{+ + -\}$ and the spectrum of the word $w' = w^{16}\{+ + +\}w^{16}$. We see that the two spectra are very close.

As is clear from this discussion, the spectrum is indeed 'incredibly complicated'.

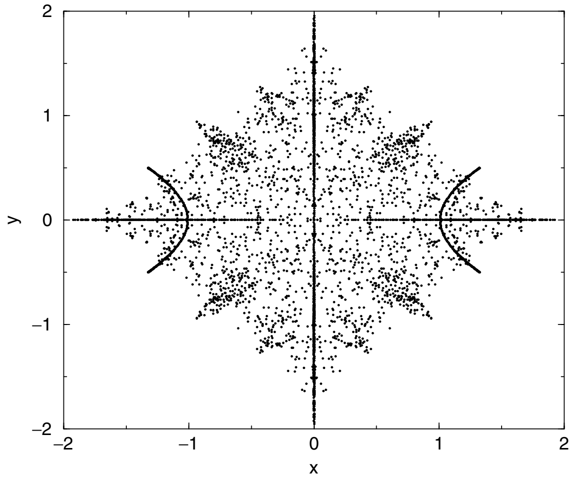

Figure 6. Support of the density of states of the word $\{+\ +\ -\}$ over the spectrum of a random 1000×1000 matrix.

6. Words and spectral curves

We content ourselves by focusing on the cuts of $\sqrt{Q(z)}$. Since for a word w of length L, $Q(z)$ is a polynomial of degree $2L$ with $2L$ roots, it gives rise to L curved line segments. The curves are given by the condition

$$\operatorname{Im} Q(z) = 0. \tag{28}$$

(The sign of Re $Q(z)$ depends on the choice for the square root branch cut.) Given a word w of length L, the corresponding spectrum must be invariant under a cyclic permutation of the letters $w_1, w_2, \ldots, w_L$, namely under $w_1 \to w_2, w_2 \to w_3, \ldots, w_L \to w_1$.

As an example, for the three-letter word $w = \{+\ +\ -\}$, $R(z) = z^2 + 1$ and $Q(z) = z^6 - 2z^4 + z^2 + 4$, which has roots at $z = \pm \mathrm{i}$, $z = \pm(\sqrt{7} - \mathrm{i})/2$, and $z = \pm(\sqrt{7} + \mathrm{i})/2$.

It is now clear what the words correspond to 'physically': a matrix H with r_k given by an endless repetition of $(+1, +1, -1)$ has a spectrum given by a straight line connecting $z = \pm\mathrm{i}$, and two algebraic curves connecting $z = (\sqrt{7} + \mathrm{i})/2$ to $z = (\sqrt{7} - \mathrm{i})/2$ and $z = -(\sqrt{7} + \mathrm{i})/2$ to $z = -(\sqrt{7} - \mathrm{i})/2$, plus poles at $z = \pm\mathrm{i}$. Note that the two poles are buried under a cut. In figure 6 we show the spectrum associated with the word $\{+\ +\ -\}$ together with the spectrum of a random 1000×1000 matrix.

We now give a complete study of all four-letter words. The polynomial is easily found to be

$$Q_4(z) = z^8 - 2sz^6 + (s^2 + 2\kappa)z^4 - 2s\kappa z^2 + \omega^2 \tag{29}$$

with $s \equiv w_1 + w_3 + w_2 + w_4$, $\kappa \equiv w_1 w_3 + w_2 w_4$ and $\omega \equiv w_1 w_3 - w_2 w_4$. Condition (28) for the curves in the spectrum reduces to

$$xy\left(y^2 - x^2 + \frac{s}{2}\right)(x^4 + y^4 - 6x^2y^2 + s(y^2 - x^2) + k) = 0. \tag{30}$$

There are only three non-trivial four-letter words, namely $w = \{+\ +\ +-\}$, $\{+\ +\ --\}$ and $\{+\ -\ --\}$. Their contribution to the spectrum of H together with the spectrum of a random 4000×4000 matrix is shown in figure 7.

In figure 8 we show the contribution of all one-, two-, three- and four-letter words to the density of states.

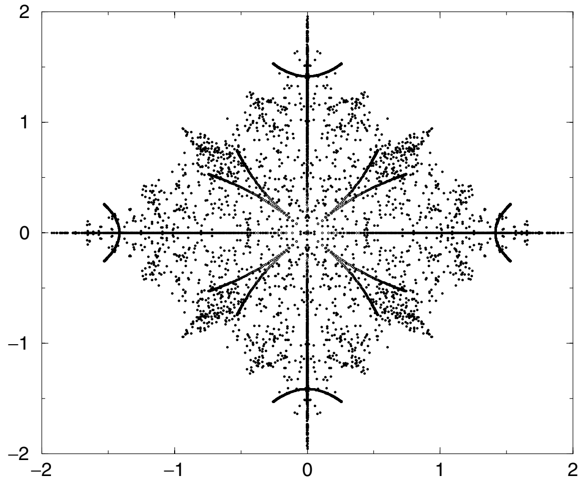

Figure 7. Support of the density of states of the periodic words $\{+++-\}$, $\{++--\}$ and $\{+---\}$ over the support of the spectrum of a random 4000×4000 matrix.

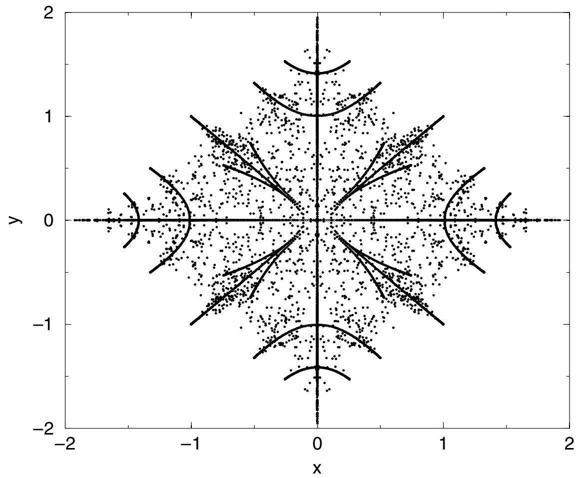

Figure 8. Support of the density of states of all periodic words of length 4 or smaller, over the support of the spectrum of a random 4000×4000 matrix.

Thus, an $N \times N$ matrix H with r_k given by repeating the word w has a spectrum determined by the stable fixed point value $b(z)$ corresponding to w. Furthermore, consider an $N \times N$ matrix H with r_k given by first repeating the word w_1 (of length L_1) N_1 times and then by repeating the word w_2 (of length L_2) N_2 times. As we would expect, in the limit in which N_1, N_2 and $N = N_1L_1 + N_2L_2$ all tend to infinity, the spectrum of H is given by superposing the spectra of H_i ($i = 1, 2$), where H_i is constructed with r_k given by first repeating the word w_i N_i times. This clearly generalizes. In figures 9–11 we show the spectrum of the word $w_1 = \{++--\}$, the spectrum of $w_2 = \{+++-\}$ and the spectrum of the word $w = w_1^{20}w_2^{20}$. We see the superposition principle at work.

From this discussion it becomes clear why the spectrum of the matrix H in FZ is so complicated. The sequence $\{r_1, r_2, \ldots, r_\infty\}$ is a book written in the binary alphabet that, in

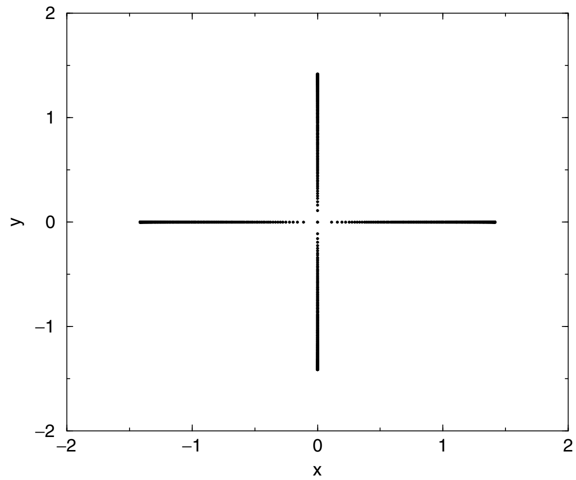

Figure 9. Support of the density of states of the periodic word $\{+\ +\ -\ -\}$.

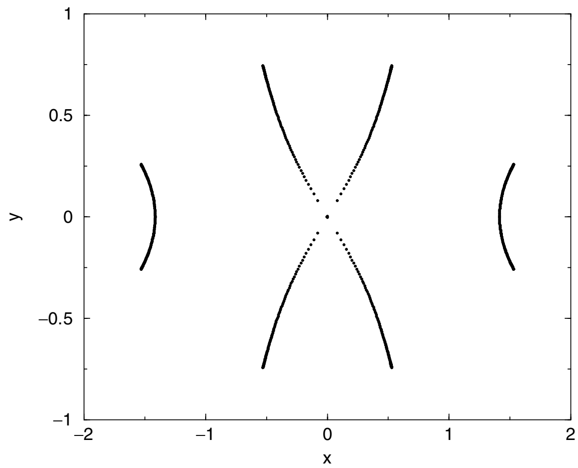

Figure 10. Support of the density of states of the periodic word $\{+\ +\ +\ -\}$.

the mathematical limit $N \to \infty$, contains all possible words, sentences and paragraphs. In fact, H contains everything ever written or that will be written in the history of the universe, including this particular paper. This familiar mind-boggling fact accounts for the complicated-looking spectrum first observed in FZ.

It also explains why numerical studies of the spectrum suggest that it is zero-dimensional. Even for N as large as 1000 the sequence contains an infinitesimally small subset of the set of all possible words, sentences and paragraphs.

7. Eigenvalues on the unit circle

In the ensemble of all books there are particularly simple books such that $\{r_1, r_2, \ldots, r_\infty\}$ consists of a word w of length L repeated again and again. In this case, we can determine the spectrum explicitly by two different methods.

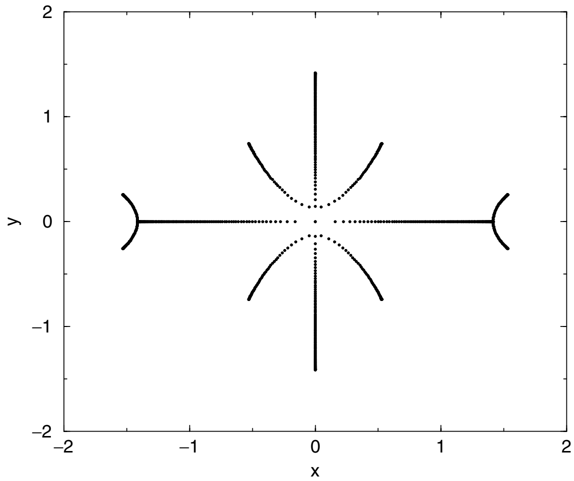

Figure 11. Support of the density of states of the composition of the two words in figures 9 and 10.

Let W be the transfer matrix corresponding to w. In other words, $W = \prod_{j=1}^{L} \begin{pmatrix} z & -w_j \\ 1 & 0 \end{pmatrix}$, where the matrix product is ordered. After repeating the word R times, we have

$$\begin{pmatrix} \Delta_{RL+1} \\ \Delta_{RL} \end{pmatrix} = W^R \begin{pmatrix} \Delta_1 \\ \Delta_0 \end{pmatrix}. \tag{31}$$

Diagonalizing $W = S^{-1}\begin{pmatrix} \lambda_1(z) & 0 \\ 0 & \lambda_2(z) \end{pmatrix} S$, we immediately see that Δ_{RL} is a linear function of $\lambda_1^R(z)$ and $\lambda_2^R(z)$:

$$\Delta_{RL} = \alpha\lambda_1^R + \beta\lambda_2^R. \tag{32}$$

We remind the reader that all quantities in (32) are functions of z.

The spectrum of H is determined by the zeros of $\Delta_{RL}(z)$ as $R \to \infty$. We note that in this limit the solution of

$$\Delta_{RL}(z) = 0 \tag{33}$$

does not depend on knowing the detailed form of α and β. Indeed, (33) implies

$$\lambda_1 = \left(-\frac{\beta}{\alpha}\right)^{\frac{1}{R}} \lambda_2 \tag{34}$$

or

$$\lambda_1^2 = \pm\left(-\frac{\beta}{\alpha}\right)^{\frac{1}{R}} \tag{35}$$

since $\det W = \lambda_1\lambda_2 = \pm 1$. In the limit $R \to \infty$, $(-\beta/\alpha)^{\frac{1}{R}}$ tends towards a (z-dependent) complex number of modulus unity. Thus, we conclude that

$$|\lambda(z)| = 1 \tag{36}$$

namely, that the eigenvalues of W lie on the unit circle. This constraint suffices to determine the eigenvalues of H. Plugging (36), that is $\lambda = \mathrm{e}^{\mathrm{i}\theta}$, into the eigenvalue equation

$$\lambda^2 - (\mathrm{tr}\, W)\lambda + \det W = 0 \tag{37}$$

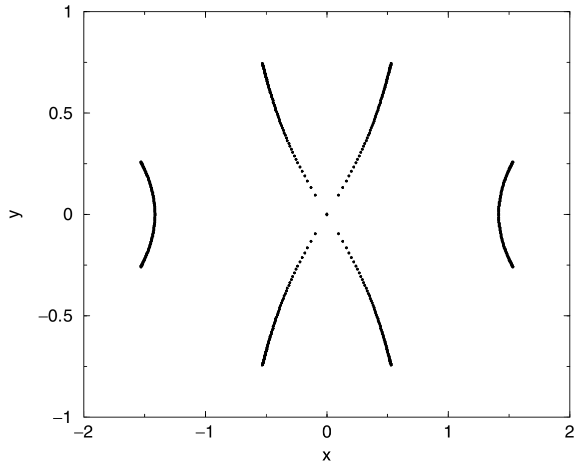

Figure 12. Support of the density of states of the word $\{+ + + -\}$ as given by equation (39).

we obtain $(\mathrm{tr}\, W) = 2\cos\theta$ if $\det W = +1$ and $(\mathrm{tr}\, W) = 2\mathrm{i}\sin\theta$ if $\det W = -1$, which we can combine into the single equation

$$(\mathrm{tr}\, W) = 2(\det W)^{\frac{1}{2}}\cos\theta \tag{38}$$

after a trivial phase shift.

As θ ranges from 0 to 2π, this traces out the spectrum in the complex plane. As an example, consider the four-letter word $\{+ + + -\}$, in which case (38) reduces to

$$z^4 - 2z^2 = 2\mathrm{i}\cos\theta. \tag{39}$$

This traces out the algebraic curve shown in figure 12, which is to be compared to figure 10.

Of course, since H is now translation invariant with period L, we can apply Bloch's theorem to determine the spectrum of H. Imposing $\psi_{k+L} = \mathrm{e}^{\mathrm{i}\varphi}\psi_k$ we reduce the eigenvalue problem of H to the eigenvalue problem of the $L \times L$ matrix

$$h_L = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & r_L\mathrm{e}^{-\mathrm{i}\varphi} \\ r_1 & 0 & 1 & \ddots & 0 & 0 \\ 0 & r_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & r_{L-2} & 0 & 1 \\ \mathrm{e}^{\mathrm{i}\varphi} & 0 & \cdots & 0 & r_{L-1} & 0 \end{pmatrix}. \tag{40}$$

One can verify that with a suitable relation between θ and φ the eigenvalue equation $\det(zI_L - h) = 0$ becomes identical to (38).

8. Conclusion and open questions

We have seen that the structure of this simple tridiagonal non-Hermitian random matrix possesses an amazing richness. This complexity can be understood if one realizes that the spectrum of the random matrix is the sum of the spectra of all tridiagonal matrices with a periodic subdiagonal obtained by repeating an infinite number of times any finite word of length L, weighted by a factor $1/2^L$.

The number of lines does not have the cardinal of the continuum. The number of lines is equal to the number of words of any length that can be made with a 2-letter alphabet; this is a countable number.

There are many open questions concerning the fine structure of the spectrum, such as whether the spectrum contains holes in the complex plane (in its domain of definition). We also have not touched upon the question of the nature of the eigenstates. Are they localized or delocalized? Numerical data seem to suggest a localization transition. We hope to address these and other questions in future work.

Acknowledgments

One of us (HO) would like to thank M Bauer, D Bernard and J M Luck for helpful discussions. This work was partially supported by the NSF under grant PHY99-07949 to the ITP.

Appendix. Cyclic invariants

As remarked in the text, the coefficients of $Q_L(z)$, an even polynomial of degree $2L$, must be constructed out of the cyclic invariants made of $w_1, w_2, \ldots, w_L$. There is presumably a well-developed mathematical theory of cyclic invariants, but what we need we can easily deduce here.

For any L, we have the two obvious cyclic invariants $s = \sum_{j=1}^{L} w_j$ and $p = \Pi_{j=1}^{L} w_j$. The number of cyclic invariants grows rapidly with L. Apparently different cyclic invariants can be constructed out of other cyclic invariants, for example $\left(\sum_{j=1}^{L} w_j\right)^2 - \left(\sum_{j=1}^{L} w_j^2\right) = 2\sum_{j\neq i}^{L} w_j w_i$.

It is easy to work out $Q_L(z)$ for low values of L, as follows:

$$Q_2(z) = z^4 - 2sz^2 + d^2 \tag{41}$$

with $d \equiv w_1 - w_2$,

$$Q_3(z) = (z^3 - sz)^2 - 4p = z^6 - 2sz^4 + s^2z^2 - 4p \tag{42}$$

where we have written $Q_3(z)$ in a form which shows that its roots can be found explicitly,

$$Q_4(z) = z^8 - 2sz^6 + (s^2 + 2\kappa)z^4 - 2s\kappa z^2 + \omega^2 \tag{43}$$

with $\kappa \equiv w_1w_3 + w_2w_4$ and $\omega \equiv w_1w_3 - w_2w_4$,

$$Q_5(z) = z^{10} - 2sz^8 + (s^2 + 2\kappa)z^6 - 2s\kappa z^4 + \kappa^2z^2 - 4p \tag{44}$$

with $\kappa \equiv w_1w_3 + w_1w_4 + w_2w_4 + w_2w_5 + w_3w_5$,

$$Q_6(z) = z^{12} - 2sz^{10} + (s^2 + 2\kappa)z^8 - 2(s\kappa + \rho)z^6 + (\kappa^2 + 2s\rho)z^4 - 2\kappa\rho z^2 + \delta^2 \tag{45}$$

where

$$\kappa = w_1w_3 + w_1w_4 + w_2w_4 + w_1w_5 + w_2w_5 + w_3w_5 + w_2w_6 + w_3w_6 + w_4w_6$$
$$\rho = w_1w_3w_5 + w_2w_4w_6 \qquad \delta = w_1w_3w_5 - w_2w_4w_6$$

and

$$Q_7(z) = z^{14} - 2sz^{12} + (s^2 + 2\kappa)z^{10} - 2(s\kappa + \rho)z^8 + (\kappa^2 + 2s\rho)z^6 - 2\kappa\rho z^4 + \rho^2z^2 - 4p \tag{46}$$

with

$$\kappa = w_1w_3 + w_1w_4 + w_2w_4 + w_1w_5 + w_2w_5 + w_3w_5 + w_1w_6 + w_2w_6 + w_3w_6 + w_4w_6 + w_2w_7 + w_3w_7 + w_4w_7 + w_5w_7$$

$$\rho = w_1w_3w_5 + w_1w_3w_6 + w_1w_4w_6 + w_2w_4w_6 + w_2w_4w_7 + w_2w_5w_7 + w_3w_5w_7.$$

The quantities $d, \kappa, \omega, \kappa, \rho, \delta$ are manifestly cyclic invariants.

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